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Rational Approximation to x^n

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The beautiful classical theory of the Tchebychev polynomials arises, as we know, from the problem of best uniform approximation to x^n on $[-1, 1]$ by polynomials of degree $<n$. Recently Reddy asked the very natural question: How do we determine the best uniform approximation to x^n on $[-1, 1]$ by *rational functions* of degree $<n$?

We address ourselves to this problem and certain slight generalizations of it. While we do not obtain the exact and explicit answers as in the Tchebychev case, we do obtain precise results on order of magnitude. For example the answer to Reddy's question turns out to be that the proximity of rational functions of degree $<n$ is of the exact order $n^{1/2}(3(3)^{1/2})^{-n}$ (as compared to $2 \cdot 2^{-n}$ in the polynomial case).

THEOREM. *Let s and n be any nonnegative integers; then*

(i) *There is a polynomial $p(x)$ of degree $<n$ and a polynomial $q(x)$ of degree $2s$ such that, throughout* $[-1, 1]$,

$$\left| x^n - \frac{p(x)}{q(x)} \right| \leqslant 2^{1-n} \binom{s+n-3}{s}^{-1}.$$

(ii) *If $p(x)$ is a polynomial of degree $<n$ and $q(x)$ is a polynomial of degree $\leqslant 2s$, then, somewhere in* $[-1, 1]$,

$$\left| x^n - \frac{p(x)}{q(x)} \right| \geqslant 2^{-2-n} \binom{s+n+1}{s}^{-1}.$$

Note that these upper and lower bounds are only separated by a factor of the order $(1 + (s/n))^4$, which is rather negligible compared to the size of the binomial coefficients involved. Indeed, if $s \leqslant cn$, then it is essentially just a constant factor (as in Reddy's original case) and we have the correct order of magnitude) answer.

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Before proving our theorem we turn to some intimately connected considerations which are of some interest in themselves.

We define the analytic part, A, of a series $f(x) = \sum_{-\infty}^{\infty} C_v x^v$ to be $A(f(x)) = \sum_{v=0}^{\infty} C_v x^v$.

Our principal concern lies in estimating the "size" of $A(p(x)/x^n)$ in terms of the size of $p(x)$. Here $p(x)$ is a polynomial, and size is measured by $\|f(x)\| = \mathrm{Max}_{-1\leqslant x\leqslant 1} |f(x)|$. The decisive tool for this job is a formula for $A(T_m(x)/x^n)$, where the $T_m(x)$ are the Tchebychev polynomials $\cos(m \cos^{-1} x)$ when $m > 0$ (but where we find it convenient to write $T_0(x) = \frac{1}{2}$). We have, then, with $N = [(m + n)/2]$,

$$F\colon A\left(\frac{T_m(x)}{x^n}\right) = (-2)^n \sum_{k=n}^{N} (-1)^k \binom{k-1}{n-1} T_{m+n-2k}(x).$$

The proof is by direct application of the generating function formula

$$\sum_{v=0}^{\infty} T_v(x)\, r^v = \frac{R}{1 - 2rx + r^2}, \qquad R = \frac{1 - r^2}{2}. \tag{1}$$

We have, namely, from (1), that

$$\sum T_v(x)\, r^v = \frac{R}{1 + r^2} \sum_{i=0}^{\infty} \left(\frac{2rx}{1 + r^2}\right)^i, \tag{2}$$

so that dividing by x^n and applying A gives

$$\sum A\left(\frac{T_v(x)}{x^n}\right) r^v = \left(\frac{2r}{1 + r^2}\right)^n \cdot \frac{R}{1 + r^2} \sum_{j=0}^{\infty} \left(\frac{2rx}{1 + r^2}\right)^j$$

$$= \left(\frac{2r}{1 + r^2}\right)^n \frac{R}{1 - 2rx + r^2}. \tag{3}$$

Using (1) again, as well as the identity

$$\frac{1}{(1 + r^2)^n} = \sum_{i=0}^{\infty} (-1)^i \binom{n + i - 1}{n - 1} r^{2i}, \tag{4}$$

Eq. (3) becomes

$$\sum A\left(\frac{T_v(x)}{x^n}\right) r^v = 2^n r^n \sum (-1)^i \binom{n + i - 1}{n - 1} r^{2i} \sum T_v(x) r^v \tag{5}$$

and F follows immediately upon comparing coefficients of r^m.

We now derive bounds for the $A(T_m(x)/x^n)$ from the formula F. First of all we have the trivial upper bound

$$\left\| A\left(\frac{T_m(x)}{x^n}\right)\right\| \leqslant 2^n \sum_{k\leqslant(m+n)/2} \binom{k-1}{n-1} = 2^n \binom{N}{n}. \tag{6}$$

As to lower bounds we employ the well-known identity

$$\begin{aligned}(1-\cos\theta)&\Big(a_0 + 2\sum_{j\geqslant 1} a_j \cos j\theta\Big)\\ &= \sum_{j\geqslant 0} (a_j - 2a_{j+1} + a_{j+2})(1-\cos(j+1)\,\theta),\end{aligned} \tag{7}$$

the validity of which, for terminating sequences, at least, follows immediately by comparing coefficients.

The point is that if $m+n$ is even and we set $x = \sin(\theta/2)$, and choose $a_j = \binom{N-1-j}{n-1}$ (N here, as always, is $[(m+n)/2]$), then the sum $a_0 + 2\sum a_j \cos j\theta$ becomes

$$(-1)^N 2 \sum (-1)^k \binom{k-1}{n-1} T_{m+n-2k}(x).$$

Furthermore, in this case,

$$a_j - 2a_{j+1} + a_{j+2} = \binom{N-3-j}{n-3}.$$

Inserting all of this into (7), therefore, gives

$$\begin{aligned}(-1)^N 2 &\sum_k (-1)^k \binom{k-1}{n-1} T_{m+n-2k}(x)\\ &= \frac{1}{(1-\cos\theta)} \sum_{j\geqslant 0} \binom{N-3-j}{n-3} (1-\cos(j+1)\,\theta)\\ &\geqslant \binom{N-3}{n-3} \qquad \text{throughout } -1 \leqslant x \leqslant 1.\end{aligned} \tag{8}$$

If (8) is now substituted into F we obtain our desired lower bound

$$\left| A\left(\frac{T_m(x)}{x^n}\right)\right| \geqslant 2^{n-1} \binom{N-3}{n-3} \qquad \text{throughout } [-1, 1],$$

provided $m+n$ is even, and $N = (m+n)/2$.

Finally we may employ (6) to obtain upper gounds for $A(P(x)/x^n)$ in general. If we write, namely

$$P(x) = \sum_{\mu=0}^{m} C_\mu T_\mu(x), \qquad m = \deg P, \quad \| P \| = 1, \tag{10}$$

then we have, for $m > 0$,

$$C_\mu = \frac{2}{\pi} \int_{-1}^{1} \frac{P(x)\, \bar{T}_\mu(x)}{(1 - x^2)^{1/2}}\, dx,$$

so that $| C_\mu | \leqslant 2$. Therefore if we apply (6) to (10) we obtain the bound

$$2^n \sum_{\mu=1}^{m} | C_\mu | \binom{[(n+\mu)/2]}{n} \leqslant 2^{n+1} \sum_{\mu=1}^{m} \binom{[(n+\mu)/2]}{n}$$

$$\leqslant 2^{n+1} 2 \cdot \sum_{j \geqslant 0} \binom{N-j}{n} = 2^{n+2} \binom{N+1}{n+1}. \tag{11}$$

Summarizing, we have, then,

$$\left\| A\left(\frac{P(x)}{x^n}\right) \right\| \leqslant 2^{n+2} \binom{N+1}{n+1} \qquad \text{for any } P \text{ with } \deg P \leqslant m, \tag{12}$$

$\| P \| \leqslant 1$, where, again, $N = [(m+n)/2]$.

We now easily give the proof of our theorem:

(i) Choose $p(x)$ and $q(x)$ so that $x^n q(x) - p(x) = T_{n+2s}(x)$. Thus $q(x) = A(T_{n+2s}(x)/x^n)$, (9) applies, and we have

$$\left\| x^n - \frac{p(x)}{q(x)} \right\| \leqslant \left[2^{n-1} \binom{N-3}{n-3} \right]^{-1} \qquad \text{where} \quad N = n + s,$$

as required.

(ii) If we call $P(x) = x^n q(x) - p(x)$ and normalize so that $\| P \| = 1$, then $q(x) = A(P(x)/x^n)$, and we obtain

$$\left\| x^n - \frac{p(x)}{q(x)} \right\| \geqslant \frac{\| x^n q(x) - p(x) \|}{\| q(x) \|} = \frac{1}{\| A(P(x)/x^n) \|}.$$

By (12) we have

$$\left\| A\left(\frac{P(x)}{x^n}\right) \right\| \leqslant 2^{n+2} \binom{n+s+1}{n+1},$$

however, and so (ii) follows.